**Module 4: Systems of Equations and Matrices**

**III. Matrices and Systems of Equations**

After completing this section, you should be able to:

* write the coefficient matrix and the augmented matrix for a system of linear equations
* solve systems of equations using matrices

**A. Matrix Notation**

Thinking back over the Gauss elimination technique used to solve 3 × 3 systems of linear equations, recall that the coefficients and constants in the equations were manipulated, and the variables *x*, *y*, and *z* were written over and over again. The elimination process can be streamlined by using a sort of shorthand notation which focuses on the coefficients and constants, and removes the need for repeatedly writing *x*, *y*, and *z*.

The shorthand notation involves the use of matrices.

A **matrix** is an ordered array of numbers arranged in rows and columns. If a matrix has *m* rows and *n* columns, it is said to be of order *m* × *n*. Here are some examples of matrices of a variety of orders:

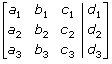
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| --- | --- | --- | --- | --- |
| 2 × 2 | 3 × 3 | 3 × 1 | 1 × 3 | 2 × 3 |
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If the number of rows of a matrix is equal to the number of columns, the matrix is *square*.

Given a system of three equations in three variables

*a*1*x* + *b*1*y* + *c*1*z* = *d*1  
*a*2*x* + *b*2*y* + *c*2*z* = *d*2  
*a*3*x* + *b*3*y* + *c*3*z* = *d*3

the square 3 × 3 matrix  is called the *coefficient matrix*, and

the 3 × 4 matrix  is called the *augmented matrix*. In an augmented matrix, the entries to the right of the vertical line in the matrix are the constants on the right side of the linear equations, and the entries to the left of the vertical line are the coefficients.

**Example III.A.1:** Write the coefficient matrix and the augmented matrix for the system

*x* – 2*y* – *z* = 4  
*y* + 3*z* = –1  
*z* = –2

**Solution:**

Some of the coefficients of this system are 0. For example, there is no *x*-term in the second equation, so the *x*-coefficient is 0.

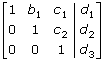
*x* - 2*y* -*z* =  4  
0*x*+ *y* + 3*z* = -1  
0*x* + 0*y* *+  z* = -2

The coefficient matrix is  and the augmented matrix is https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M4-Module_4/images/Mod4-10b.gif.

**B. Solutions of Systems of Equations using Matrix Notation**

Recall that a system of the triangular form below is easy to solve using back-substitution.

*x* + *b*1*y* + *c*1*z* = *d*1  
*y* + *c*2*z* = *d*2  
*z* = *d*3

The augmented matrix for this system is .

Note the triangular form of the nonzero coefficients.

Gauss elimination using matrices follows the same basic process as Gauss elimination using equations. Given a system of linear equations, the goal is to transform the augmented matrix into an equivalent matrix in triangular form, and then use back-substitution to find the solution.

In the previous topic, you manipulated equations in order to carry out Gauss elimination. Now you will be manipulating rows of the augmented matrix.

In Gauss elimination using matrices, there are three operations which are employed to manipulate rows of the augmented matrix:

1. **Interchange:** Switch two rows of the matrix.
2. **Scale:** Multiply each entry in a row by a nonzero scalar (a nonzero constant).
3. **Replace:** Replace a row by the sum of the row and a nonzero multiple of another row. For example, you could replace the 2nd row by [5 · (1st row) + 2nd row].

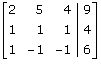
To see how the process works, consider the same system as in example II.C.1. If you review example II.C.1, you will see that every step in equation manipulation has a counterpart in row manipulation.

**Example III.B.1:** Solve the following system using Gauss elimination with matrices.

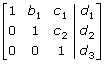
2*x* + 5*y* + 4*z* =9  
*x* + *y* + *z* = 4  
*x* - *y* - *z* = 6

**Solution:**

Write the augmented matrix for the system:



The goal is to manipulate this matrix and arrive at an equivalent matrix in triangular form

,

with leading nonzero coefficients of 1 in each row, and then use back-substitution to determine the solution.

**Step 1:** Start by manipulating the augmented matrix so that the row 1, column 1 entry is 1.

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The first row does not require any further modification. However, in order to arrive at triangular form, in the first column, the entries below the 1 entry must be 0.

**Step 2:** Get a 0 entry in the row 2, column 1 position.

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**Step 3:** Get a 0 entry in the row 3, column 1 position.

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The first column matches the desired triangular form.

Next, manipulate the matrix so that the leading nonzero entry in the 2nd row or 3rd row is equal to 1. You can choose to work with the second row or the third row.

If the second row is multiplied by 1/3, the resulting row will have entries 0, 1, 2/3, and1/3.

If the third row is multiplied by –1/2, the resulting row will have entries 0, 1, 1 and –1.

Because it is easier to perform arithmetic involving integers rather than fractions, it is preferable to work with the third row.

**Step 4:** Get a 1 entry in the row 3, column 2 position.

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In the triangular form, the entry in row 2, column 2 should be 1, so switch the last two rows.

**Step 5:** Get a 1 entry in the row 2, column 2 position.

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The second row does not require any further modification. However, in order to arrive at triangular form, in the second column, the entry below the 1 entry must be 0.

**Step 6:** Get a 0 entry in the row 3, column 2 position.

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Now manipulate the system so that the third row has a leading nonzero coefficient of 1.

**Step 7:** Get a 1 entry in the row 3, column 3 position.

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The system now has the desired triangular form.

**Step 8:** Use back-substitution to determine the solution of the triangular system.

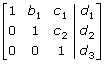
Write the system of equations corresponding to the final augmented matrix .

*x* + *y* + *z* =  4  
*y* + *z* = -1  
*z* = -4

According to the third equation, *z* = –4. Use back-substitution to find *y* and then *x*.

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| --- | --- |
| *y* +     *z* = –1 | Use the second equation. |
| *y* + (–4) = –1 | Substitute for *z*. |
| *y*= 3 | Solve for *y*. |
|  |  |
| *x* +   *y* +     *z*  = 4 | Use the first equation. |
| *x* + (3) + (–4) = 4 | Substitute for *y* and *z*. |
| *x*                    = 5 | Solve for *x*. |

The solution (*x*, *y*, *z*) = (5, 3, –4). The solution may be verified by checking to see that it satisfies all three equations.

If the end result of Gauss elimination is a matrix of the form , the system of equations must have exactly one solution, which can be found by back-substitution. Notice the triangular pattern of nonzero entries in the coefficient matrix.

If Gauss elimination results in a row of 0's except for a 1 in the last column, that row corresponds to an equation of the form 0 = 1. The system of linear equations is inconsistent and has no solution.

For example, the third row of the augmented matrix  corresponds to the equation 0 = 1, which is never true, so the system has no solution.

If Gauss elimination does not result in an inconsistent system, but there is a row consisting of all 0's, then that row corresponds to an equation of the form 0 = 0. The system of linear equations has infinitely many solutions.

For example, the third row of the augmented matrix  corresponds to the equation 0 = 0, so the system has infinitely many solutions.

For each of these possibilities (one solution, no solution, infinitely many solutions), the final matrix produced by Gauss elimination has a form called *row-echelon form*, defined in detail below.

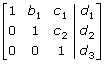
**Row-Echelon Form**

A matrix is in *row-echelon form* if all of the following conditions are satisfied:

1. The first nonzero number in each row is a 1 (called a *leading*1).
2. As you go from one row to the next, the leading 1 is located farther to the right. For example, in a 4 × 5 matrix, if the leading 1 in row 2 is in column 3, then a leading 1 in row 3 cannot appear in columns 1, 2, or 3; it may only appear farther to the right, in columns 4 or 5.
3. All 0 rows (rows whose entries are all 0) are located at the bottom of the matrix.

A matrix is in *reduced row-echelon form* if conditions 1, 2, and 3 are satisfied, and

1. If a column contains a leading 1, then all of the other entries in that column are 0.

The matrices , , and  are all in row-echelon form.

The matrix  is not in row-echelon form because the first nonzero entry in row 2 is not a 1. (It is perfectly acceptable for the second entry in row 1 to be a 0.)

The matrices , , and https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M4-Module_4/images/Mod4-34c.gif are in reduced row-echelon form, but  is not in reduced row-echelon form, because the third column contains a leading 1 and another nonzero entry (the 5 in row 1).

With Gauss elimination, row operations are performed until a matrix in row-echelon form is generated, and then back-substitution is used. If more row operations are performed to obtain reduced row-echelon form, the method of elimination is called *Gauss-Jordan elimination*, named for two mathematicians.

**Example III.B.2:** Solve the following system using Gauss-Jordan elimination.

2*x* + 5*y* + 4*z* = 9  
*x* + *y* + *z* = 4  
*x* - *y* - *z* = 6

**Solution:**

In example III.B.1, Gauss elimination produced a matrix in row-echelon form:



The goal is to obtain a matrix in reduced row-echelon form. In order to be in this form, the entries above the leading 1's must be 0.

**Step 1:** Get a 0 entry in the row 2, column 3 position.

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**Step 2:** Get a 0 entry in the row 1, column 3 position.

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**Step 3:** Get a 0 entry in the row 1, column 2 position.

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Now the matrix is in reduced row-echelon form.

**Step 4:** Find the solution.

The matrix  corresponds to the system of equations

*x* = 5  
*y* = 3  
*z* = -4

The solution is the ordered triple (5, 3, –4). This is the same answer obtained using Gauss elimination and back-substitution in example III.B.1. The advantage of the Gauss-Jordan method is that the back-substitution phase is avoided by performing additional row operations.

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